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2005 J. Phys. A: Math. Gen. 38 10101

(http://iopscience.iop.org/0305-4470/38/47/004)

J. Phys. A: Math. Gen. **38** (2005) 10101–10106 doi:10.1088/0305-4470/38/47/004

Perturbed Hankel determinants

Estelle Basor[1] **and Yang Chen**[2,3]

[1] Department of Mathematics, Calpoly, San Luis Obispo, CA, USA
[2] Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, People's Republic of China

E-mail: ebasor@calpoly.edu and ychen@ic.ac.uk

Received 15 September 2005
Published 9 November 2005
Online at stacks.iop.org/JPhysA/38/10101

Abstract

In this paper, we compute, for large n, the determinant of a class of $n \times n$ Hankel matrices, which arise from a smooth perturbation of the Jacobi weight. For this purpose, we employ the same idea used in previous papers, where the unknown determinant $D_n[w_{\alpha,\beta}h]$ is compared with the known determinant $D_n[w_{\alpha,\beta}]$. Here $w_{\alpha,\beta}$ is the Jacobi weight and $w_{\alpha,\beta}h$, where $h = h(x), x \in [-1, 1]$, is strictly positive and real analytic, is the smooth perturbation on the Jacobi weight $w_{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta}$. Applying a previously known formula on the distribution function of linear statistics, we compute the large-n asymptotics of $D_n[w_{\alpha,\beta}h]$ and supply a missing constant of the expansion.

PACS number: 02.10.Yn

1. Introduction and preliminaries

The purpose of this paper is to find heuristically an asymptotic expansion for determinants of certain Hankel matrices. The matrices are generated by the moments of a function defined on the interval $[-1, 1]$. Let $w(x)$ be a function of the form

$$w_{\alpha,\beta}(x)h(x),$$

where

$$w_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}, \qquad \alpha \geqslant 0, \quad \beta \geqslant 0,$$

and $h(x)$ is a strictly positive function with a derivative satisfying a Lipschitz condition.

Define

$$\mu_k[w] = \int_{-1}^{1} x^k w(x)\,\mathrm{d}x, \qquad k = 0, 1, 2, \ldots,$$

[3] On leave from Imperial College London.

and

$$D_n[w] = \det(\mu_{j+k}[w])_{j,k=0}^{n-1}.$$

The motivation for investigating such perturbed Hankel determinants comes from random matrix theory and its applications, where one studies the generating functions of linear statistics [4, 5]. Also see [1] and some of the references in that volume.

Our goal will be to show formally that, for $w = w_{\alpha,\beta}h$,

$$D_n[w] \sim 2^{-n(n+\alpha+\beta)} n^{(\alpha^2+\beta^2)/2-1/4} (2\pi)^n \exp\left(\frac{n}{\pi}\int_{-1}^{1}\frac{\ln h(x)}{\sqrt{1-x^2}}\,\mathrm{d}x\right) C, \qquad (1.1)$$

where the n independent constant C is given by

$$\exp\left[\frac{1}{4\pi^2}\int_{-1}^{1}\frac{\ln h(x)}{\sqrt{1-x^2}}\left(P\int_{-1}^{1}\frac{\sqrt{1-y^2}}{y-x}\frac{h'(y)}{h(y)}\,\mathrm{d}y\right)\mathrm{d}x\right]$$
$$\times \exp\left(\frac{\alpha+\beta}{2\pi}\int_{-1}^{1}\frac{\ln h(x)}{\sqrt{1-x^2}}\,\mathrm{d}x\right)\frac{G^2\left(\frac{\alpha+\beta+1}{2}\right)G^2\left(\frac{\alpha+\beta}{2}+1\right)\Gamma\left(\frac{\alpha+\beta+1}{2}\right)}{G(\alpha+\beta+1)G(\alpha+1)G(\beta+1)}.$$

In the above formula the function G is the Barnes G-function, an entire function that satisfies the difference equation $G(z+1) = \Gamma(z)G(z)$, with $G(1) = 1$. The result (1.1) is also valid for $\alpha \geqslant -1/2$, and $\beta \geqslant -1/2$, since this expression is real analytic in α and β.

The main idea, which can be traced back to a paper of Szegő [9], is that one can find the above formula in two steps. The first is to consider the 'pure' weight

$$w_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}.$$

Using some basic results from the theory of orthogonal polynomials, the Hankel determinant for the pure weight can be found exactly and then easily computed asymptotically. This step is rigorous and in fact may be the first instance where these asymptotics are found completely.

The next step is to use the linear statistics formula derived from the Coulomb fluid approach [4, 5]—expected to be valid for sufficiently large n—to heuristically compute the quotient

$$\frac{D_n[w_{\alpha,\beta}h]}{D_n[w_{\alpha,\beta}]}, \qquad (1.2)$$

thus achieving the desired result.

We note that in a recent work [8], the asymptotic formula for D_n appears, but without the constant term. In future work, we hope to use the techniques of [2] to make the ideas presented here complete and thus firmly establish the validity of the asymptotic formula.

We begin with some notation. Let $P_n(x)$ be monic polynomials of degree n in x and orthogonal, with respect to a weight, $w(x)$, $x \in [a, b]$;

$$\int_a^b P_m(x)P_n(x)w(x)\,\mathrm{d}x = h_n[w]\delta_{m,n}, \qquad (1.3)$$

where $h_j[w]$ is the square of the L^2 norm of the polynomials orthogonal with respect to w, over $[-1, 1]$.

From the orthogonality condition there follows the recurrence relation

$$zP_n(z) = P_{n+1}(z) + \alpha_n P_n(z) + \beta_n P_{n-1}(z), \qquad n = 0, 1, \ldots, \qquad (1.4)$$

where $\beta_0 P_{-1}(z) := 0$, α_n, $n = 0, 1, 2, \ldots$, is real and $\beta_n > 0$, $n = 1, 2, \ldots$.

There is an intimate relationship between the values of β_n, h_n and the Hankel determinants.

Indeed, the determinant, for any weight w,

$$D_n[w] = \prod_{j=0}^{n-1} h_j[w].$$

In addition,

$$h_j[w] = h_0[w] \prod_{k=1}^{j} \beta_k.$$

Thus, if we can compute β_j it follows that both h_j and D_n can be explicitly determined. For this and all other basic results see [10].

For the monic Jacobi polynomials, which is in the case when $w = w_{\alpha,\beta}$, it is well known that

$$\alpha_n = \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$$

and

$$\beta_n = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)(2n+\alpha+\beta-1)}.$$

Hence, it follows that

$$h_n[w_{\alpha,\beta}] = 2^{2n+\alpha+\beta+1} \frac{\Gamma(n+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)[\Gamma(2n+\alpha+\beta+1)]^2}, \tag{1.5}$$

and

$$D_n[w_{\alpha,\beta}] = 2^{-n(n+\alpha+\beta)}(2\pi)^n \frac{\Gamma\left(\frac{\alpha+\beta+1}{2}\right)G^2\left(\frac{\alpha+\beta+1}{2}\right)G^2\left(\frac{\alpha+\beta}{2}+1\right)}{G(\alpha+\beta+1)G(\alpha+1)G(\beta+1)} \times \frac{G(n+1)G(n+\alpha+1)G(n+\beta+1)G(n+\alpha+\beta+1)}{G^2\left(n+\frac{\alpha+\beta+1}{2}\right)G^2\left(n+\frac{\alpha+\beta}{2}+1\right)\Gamma\left(n+\frac{\alpha+\beta+1}{2}\right)}, \tag{1.6}$$

where $G(z)$ is the Barnes G-function. See [6] for a first-principles derivation of the recurrence coefficients.

The asymptotics of the Gamma function and the Barnes G-function are well understood. We have that

$$\Gamma(n+a) \sim \sqrt{2\pi}\,\mathrm{e}^{-n} n^{n+a-1/2},$$
$$G(n+a+1) \sim n^{(n+a)^2/2-1/12}\,\mathrm{e}^{-3n^2/4-an}(2\pi)^{(n+a)/2}K,$$

where

$$K := G^{2/3}(1/2)\pi^{1/6}2^{-1/36}.$$

From the above asymptotic expressions, an easy computation shows that

$$D_n[w_{\alpha,\beta}] \sim 2^{-n(n+\alpha+\beta)} n^{(\alpha^2+\beta^2)/2-1/4}(2\pi)^n \frac{G^2\left(\frac{\alpha+\beta+1}{2}\right)G^2\left(\frac{\alpha+\beta}{2}+1\right)\Gamma\left(\frac{\alpha+\beta+1}{2}\right)}{G(\alpha+\beta+1)G(\alpha+1)G(\beta+1)}. \tag{1.7}$$

The above formula is the promised result for the 'pure' weight. When $\alpha = 0 = \beta$, we find

$$D_n[w_{0,0}] \sim \frac{\pi^n}{n^{1/4}} \frac{1}{2^{n(n-1)}} G^2(1/2)\Gamma(1/2).$$

This is consistent with Hilbert's [7] asymptotic expression for large n, of the Hankel determinant associated with the Legendre weight,

$$(D_n[w_{0,0}])^{1/n} = \frac{\pi}{2^{n-1}}(1+\varepsilon_n), \qquad \text{where} \quad \lim_{n\to\infty} \varepsilon_n = 0,$$

and we have changed the notation of [7] to be compatible with ours.

2. Perturbed Jacobi weight

In this section we show how to compare the unknown Hankel determinant $D_n[w_{\alpha,\beta}h]$, with the known Hankel determinant $D_n[w_{\alpha,\beta}]$. It is known from [10] (see also [4, 5]) that

$$\frac{D_n[w_{\alpha,\beta}h]}{D_n[w_{\alpha,\beta}]} = \left\langle \prod_{k=1}^{n} h(x_j) \right\rangle, \tag{2.1}$$

where $\Psi = \Psi(x_1, \ldots, x_n)$, and

$$\langle \Psi \rangle := \frac{\int_{-1}^{1} \cdots \int_{-1}^{1} \Psi \prod_{1\leqslant j<k\leqslant n} (x_k - x_j)^2 \prod_{l=1}^{n} w_{\alpha,\beta}(x_l)\, \mathrm{d}x_l}{\int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{1\leqslant j<k\leqslant n} (x_k - x_j)^2 \prod_{l=1}^{n} w_{\alpha,\beta}(x_l)\, \mathrm{d}x_l}. \tag{2.2}$$

This can be rewritten as an average of the exponential of the linear statistics $\sum_{l=1}^{n} \ln h(x_l)$, i.e.,

$$\left\langle \exp\left(\sum_{l=1}^{n} \ln h(x_l) \right) \right\rangle.$$

Note that, because of the assumptions on h, $\ln h$ is well defined for $x \in [-1, 1]$. Results, at least in a heuristic way, are known about such linear statistics. In particular, the logarithm of (2.1) is, for large n,

$$\frac{1}{4\pi^2} \int_{a_n}^{b_n} \frac{\ln h(x)}{\sqrt{(b_n - x)(x - a_n)}} \left(P \int_{a_n}^{b_n} \frac{\sqrt{(b_n - y)(y - a_n)}}{y - x} \frac{h'(y)}{h(y)}\, \mathrm{d}y \right) \mathrm{d}x + \int_{a_n}^{b_n} \ln h(x) \sigma(x)\, \mathrm{d}x, \tag{2.3}$$

where the equilibrium density $\sigma(x)$, defined for $x \in [a_n, b_n]$, is

$$\sigma(x) = \frac{\sqrt{(b_n - x)(x - a_n)}}{2\pi^2} \int_{a_n}^{b_n} \frac{\mathsf{v}'(x) - \mathsf{v}'(y)}{x - y} \frac{\mathrm{d}y}{\sqrt{(b_n - y)(y - a_n)}},$$

and

$$\mathsf{v}'(x) := -\frac{w'_{\alpha,\beta}(x)}{w_{\alpha,\beta}(x)} = -\frac{\alpha}{x - 1} - \frac{\beta}{x + 1}.$$

The end points, a_n and b_n, of the support are determined by

$$2\pi n = \int_{a_n}^{b_n} \frac{x\mathsf{v}'(x)}{\sqrt{(b_n - x)(x - a_n)}}\, \mathrm{d}x, \qquad 0 = \int_{a_n}^{b_n} \frac{\mathsf{v}'(x)}{\sqrt{(b_n - x)(x - a_n)}}\, \mathrm{d}x.$$

For the derivation of (2.3) using a 'small fluctuations' approach see [1].

In our problem, the above equations become

$$n + \left(\frac{\alpha + \beta}{2} \right) = \frac{\alpha}{2\sqrt{(1 - a_n)(1 - b_n)}} + \frac{\beta}{2\sqrt{(1 + a_n)(1 + b_n)}}$$
$$0 = \frac{\alpha}{\sqrt{(1 - a_n)(1 - b_n)}} - \frac{\beta}{\sqrt{(1 + a_n)(1 + b_n)}},$$

and the solutions are

$$a_n = \frac{\beta^2 - \alpha^2 - 4\sqrt{n(n + \alpha)(n + \beta)(n + \alpha + \beta)}}{(2n + \alpha + \beta + 2)^2}$$
$$b_n = \frac{\beta^2 - \alpha^2 + 4\sqrt{n(n + \alpha)(n + \beta)(n + \alpha + \beta)}}{(2n + \alpha + \beta + 2)^2}.$$

In the Coulomb fluid approximations [3], the diagonal ($\tilde{\alpha}_n$) and off-diagonal recurrence coefficients ($\tilde{\beta}_n$) are

$$\tilde{\alpha}_n = \frac{b_n + a_n}{2} = \frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta)^2}, \qquad \tilde{\beta}_n = \frac{(b_n - a_n)^2}{16} = \frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta)^4},$$

and the deviations from the exact results are

$$\tilde{\alpha}_n - \alpha_n = \frac{\beta^2 - \alpha^2}{4n^3} + \mathrm{O}\left(\frac{1}{n^4}\right), \qquad \tilde{\beta}_n - \beta_n = -\frac{1}{16n^2} + \mathrm{O}\left(\frac{1}{n^3}\right).$$

For later reference we also note that

$$1 + a_n = \frac{\beta^2}{2n^2} + \mathrm{O}\left(\frac{1}{n^3}\right) \qquad 1 - b_n = \frac{\alpha^2}{2n^2} + \mathrm{O}\left(\frac{1}{n^3}\right).$$

A simple calculation shows that, for $x \in [a_n, b_n]$,

$$\begin{aligned}\frac{\sigma(x)}{\sqrt{(b_n - x)(x - a_n)}} &= \frac{1}{2\pi}\left[\frac{\alpha}{\sqrt{(1 - a_n)(1 - b_n)}(1 - x)} + \frac{\beta}{\sqrt{(1 + a_n)(1 + b_n)}(1 + x)}\right] \\ &= \frac{1}{\pi}\left(n + \frac{\alpha + \beta}{2}\right)\frac{1}{1 - x^2}, \qquad -1 < a_n < b_n < 1,\end{aligned}$$

where we have used

$$\frac{\alpha}{\sqrt{(1 - a_n)(1 - b_n)}} = n + \frac{\alpha + \beta}{2}, \qquad \frac{\beta}{\sqrt{(1 + a_n)(1 + b_n)}} = n + \frac{\alpha + \beta}{2}.$$

Therefore, for $x \in (-1, 1)$, and n large,

$$\sigma(x) = \frac{n + (\alpha + \beta)/2}{\pi\sqrt{1 - x^2}} + \mathrm{O}\left(\frac{1}{n}\right).$$

Put $f(x) = \ln h(x)$, and $x = R_n + r_n t$, where $R_n := (b_n + a_n)/2$, and $r_n := (b_n - a_n)/2$, the second term of (2.3) becomes

$$\left(n + \frac{\alpha + \beta}{2}\right) r_n^2 \int_{-1}^{1} \frac{f(R_n + r_n t)}{1 - (R_n + r_n t)^2}\sqrt{1 - t^2}\,\mathrm{d}t,$$

while the first term of (2.3) reads

$$\frac{r_n}{4\pi^2}\int_{-1}^{1}\frac{f(R_n + r_n s)}{\sqrt{1 - s^2}}\left(P\int_{-1}^{1}\frac{\sqrt{1 - t^2}}{t - s} f'(R_n + r_n t)\,\mathrm{d}t\right)\mathrm{d}s.$$

Now, since

$$R_n = \frac{\beta^2 - \alpha^2}{4}\frac{1}{n^2} + \mathrm{O}\left(\frac{1}{n^3}\right) \qquad r_n = 1 - \frac{\alpha^2 + \beta^2}{4}\frac{1}{n^2} + \mathrm{O}\left(\frac{1}{n^3}\right)$$

we see that the second term of (2.3) is asymptotic to

$$\left(n + \frac{\alpha + \beta}{2}\right)\int_{-1}^{1}\frac{\ln h(x)}{\pi\sqrt{1 - x^2}}\,\mathrm{d}x + \mathrm{o}(1),$$

while the first term of (2.3) is asymptotic to

$$\frac{1}{4\pi^2}\int_{-1}^{1}\frac{\ln h(x)}{\sqrt{1 - x^2}}\left(P\int_{-1}^{1}\frac{\sqrt{1 - y^2}}{y - x}\frac{h'(y)}{h(y)}\,\mathrm{d}y\right)\mathrm{d}x + \mathrm{o}(1).$$

The above two expressions combined with (1.7) give formula (1.1).

Acknowledgments

The first author is supported in part by NSF grants DMS-0200167 and DMS-0500892, and the second author by EPSRC grant EP/C534409/01.

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